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LETTER TO THE EDITOR

Dynamics of an impurity in a 1D periodic Burgers flow

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Abstract. We discuss the motion of an impurity advected by a non-uniform, unsteady flow provided by the 1D Burgers equation with initial sinusoidal velocity. The analysis mainly concerns the interaction of the impurity with the developing shock structure. Our numerical experiments suggest that the shock centre is the final equilibrium position for any impurity, independent of its properties and initial conditions. For the cases with small relaxation time the numerical results are in agreement with those obtained by a steady simplified analysis.

The motion of impurities in unsteady flows has many counter-intuitive aspects, and its investigation is of great relevance to the use of tracers for flow visualization and measurement.

A fundamental aspect of the problem is the choice of the mathematical model for the fluid velocity field (see, for instance [1–4]); in this letter we adopt the one-dimensional field provided by the Burgers equation. It is well known that this equation has been introduced by Burgers to model, in one dimension, some features of the complex phenomenon of turbulence, which are now referred to as *Burgerlence* [5].

Here the term *impurities* identifies particles (denser than the fluid) or bubbles (lighter than the fluid), which can be distinguished from the surrounding fluid and maintain their identity during the motion, typically because of the surface tension.

For the equation of motion we use a simple form, which retains the main effects of inertia and viscous drag; for more detail we refer to [6–8].

The *inertial force* exerted by a fluid of density ρ_f on a sphere of radius a and density ρ_p is given by [8]

$$F_I = \rho_f V \left[(1 + C_\mathrm{M}) \frac{\mathrm{d}u}{\mathrm{d}t} - C_\mathrm{M} \frac{\mathrm{d}v}{\mathrm{d}t} \right] \tag{1}$$

where V is the volume of the impurity ($V = 4\pi a^3/3$) and v its velocity. C_M is the added-mass coefficient, which can be exactly computed for low Reynolds number flows and spheric bodies [9]; we shall use $C_\mathrm{M} = 0.5$ for all cases considered. Moreover we adopt

$$\frac{\mathrm{d}u}{\mathrm{d}t} \to \left[\frac{\partial u}{\partial t} + (u \cdot \nabla) u(x, t) \right]_{x=R} \tag{2}$$

$R(t)$ being the body's location.

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The *drag force* is expressed by the Stokes formula

$$F_S = -\mu[v - u] \qquad \mu = 6\pi a\nu\rho_f \tag{3}$$

where ν is the kinematic viscosity of the fluid. This formula applies to small impurities or to low velocity flows, and is usually employed in most prototype studies [10]. By defining the following parameters

$$\delta = \frac{\rho_f}{\rho_p} \qquad \delta' = \frac{\delta(1 + C_M)}{1 + \delta C_M} \qquad \mu' = \frac{\mu}{\rho_p V(1 + \delta C_M)} \tag{4}$$

the equation of motion reads

$$\frac{dv}{dt} = \delta' \frac{du}{dt} - \mu'(v - u). \tag{5}$$

The relevant time scale for the motion of an impurity is the *relaxation time* t_R, i.e. the time needed to reduce by a factor e the initial velocity difference $v_0 - u_0$ between impurity and fluid; from (5)

$$t_R = \frac{1}{\mu'}. \tag{6}$$

As a mathematical model of the fluid velocity field we assume a particular solution of the 1D Burgers equation,

$$\frac{du}{dt} \equiv \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \nu > 0 \tag{7}$$

which is expected to provide large gradients. Our solution of (7) is chosen to satisfy the following initial condition

$$u(x, 0) = u_0(x) = -\pi \sin(\pi x) \tag{8}$$

where we have restricted our attention to the range $-1 \leqslant x \leqslant +1$. The Reynolds number associated to this initial flow turns out to be $Re_0 = 2\pi/\nu$. In the following computations we assume $Re_0 = 400$ which implies $\nu = 5 \times 10^{-3}\pi$.

If viscosity were absent the formation of a steady shock would be expected in $x = 0$ at a critical time $t_*^0 = 1/\pi^2 \approx 0.1$. In the presence of viscosity the formation of a true shock is inhibited. However, due to the relatively large value of the Reynolds number, the negative slope of the initial field increases up to a time t_*, comparable with t_*^0, and then decreases. The steepening of the profile is accompanied by a slow attenuation of the amplitude and corresponds to the almost inviscid (nonlinear) behaviour of the solution; the successive flattening is accompanied by a remarkable and fast decay in amplitude. We agree to refer to t_* as the viscous critical time and to the profile of the solution in a sufficiently small time interval around t_* as the *viscous shock*.

The velocity field resulting from (7)–(8) has been evaluated numerically [11] using the general form of the solution, well known in the literature (see, for instance [12, 13]), expressed as a convolution integral. In this case the critical time turns out to be $t_* \approx 0.16$ in correspondence of which the *shock slope* $m(t) = u_x(0, t)$ attains its minimum value, $m_* \approx -296$. The shock amplitude decays slowly for $t < t_*$ (remember that the inviscid solution suggests a constant amplitude) while for larger times it reduces at a rate consistent with a diffusive process.

We are primarily interested in the interaction of an impurity with the viscous shock, where the largest velocity gradients occurs. This interaction will be characterized by the parameter [14]

$$\xi = \frac{t_R}{t_*}. \tag{9}$$

Small values of ξ indicate impurities which adjust to the velocity field quite rapidly and are expected to follow the fluid motion accurately. On the contrary, large values of ξ mean that the time of response of the impurity is longer with respect to the viscous shock development, so that its behaviour is expected to be only weakly correlated with the instantaneous evolution of the velocity field.

The behaviour of the impurity will be analysed through the parameters a and δ; in figure 1 lines of constant ξ are reported on the (δ, a) plane, showing that qualitatively similar interaction is expected to occur for particles and bubbles, provided that the ratio between time scales is equal. Of course, details of the motion will be different, depending on the values of the characteristic parameters.

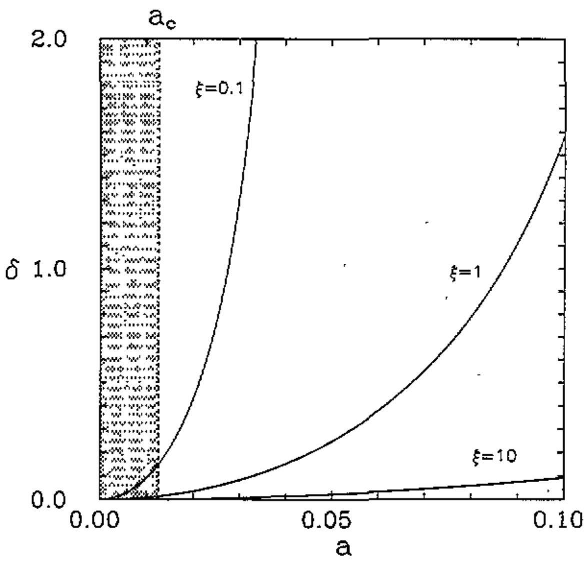

Figure 1. Lines of constant ξ in the (a, δ) plane: the shaded area indicates the stability region, $a < a_c$, from the simplified analysis.

The equation of motion (5) has been numerically integrated on a grid of 512 points. The velocity value at the instantaneous impurity position and its derivative have been estimated by cubic spline interpolation, using the Akima method [15]. Different initial conditions and parameter values have been assumed to investigate the main features of the solutions.

The effect of varying the parameter ξ is shown in figure 2 for particles released at the same initial conditions. The particle with $\xi = 0.1$ moves towards the viscous shock and reaches it at a time comparable to t_*; the local slope of the profile attains its largest (negative) value at that time and the particle is pushed away from the viscous shock centre (in fact it crosses the shock, attaining a large negative velocity). Up to times of about $2t_*$, the particle stabilizes, with spiralling motion, in a position x_M about midway between the shock centre and the point of maximum velocity. After such time, the particle moves slowly towards the shock centre, reaching it at about $4t_*$.

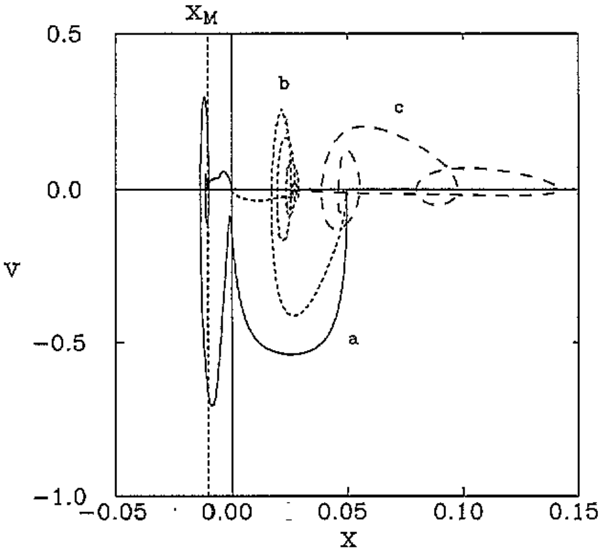

Figure 2. Trajectories in the phase plane starting from (0.05, 0): line a—$\xi = 0.1$, $\delta = 0.6$, $a = 0.02$; line b—$\xi = 1$, $\delta = 0.25$, $a = 0.05$; line c—$\xi = 10$, $\delta = 0.43$, $a = 0.2$.

Increasing the value of ξ up to 1 gives rise to a qualitative similar motion, although the particle does not cross the shock. The interaction occurs at about the same time as above, but now the response of the particle to the fluid is slower, so that it reaches the shock centre after about $13t_*$. A phase of spiralling motion is still present, at a different point, lasting up to about $6t_*$. A further increase of ξ to 10 leads to a longer time ($> 150t_*$) in approaching the stable point, with few oscillations.

The behaviour of the impurity with small ξ can be interpreted in terms of a simplified analysis, based on the idea that the velocity field approximately acts on the impurity as if it were steady, since the response time of the impurity is much smaller than the scale of evolution of the velocity field [16]. By idealizing the velocity field by means of straight lines of slope m independent of time, it results that the motion is unstable if $m(m\delta' + \mu') > 0$. For m positive (outside of the shock) the motion is always unstable, whereas for m negative (near the shock centre) the motion is unstable if $|m| > \mu'/\delta'$. Thus, for a given impurity a critical slope $m_c \equiv -\mu'/\delta'$ exists; similarly, given the Reynolds number of the flow, which limits the maximum slope, a critical radius may be defined as

$$a_c \equiv 3\sqrt{\frac{\nu}{2(1 + C_M)|m|}} \tag{10}$$

so that the motion of smaller impurities is always stable. The stability region is shaded in figure 1.

The motion of the particle with $\xi = 0.1$ is in agreement with the above simplified analysis. At the beginning of the motion the particle lies in a region of the velocity field with local slope $m(x, t)$ so that $m_c < m < 0$, and it is driven towards the centre of the shock; at $t \approx 0.14 \approx t_*$, m becomes smaller than m_c, and the particle is rejected; after a sufficiently long time, m increases to above m_c, and the shock centre is still attractive.

We observe that the position at which the particle stabilizes is such that the local slope assumes values near to m_c. A numerical integration using a velocity profile $u(x, t) = m(t)x$ in the entire $(-1, 1)$ range shows that the local curvature of the velocity profile is responsible for the spiralling motion. In this case, a particle released at the same initial condition approaches the point $x = 0$ and is then rejected when the slope attains the critical value, but no spiralling motion occurs.

This explanation cannot be extended to the case of higher ξ, because the time variation of the profile cannot be neglected; however, some common features appear, as can be noted in figure 2.

The effect of different initial conditions has also been investigated. In figure 3 we show the results of varying x_0 keeping fixed $v_0 = 0$, for the case $\xi = 0.1$ and $a = 0.02$. The position of stability $x_M \sim -0.01$ is common to the particles crossing the shock at a time for which $m < m_c$; on the contrary, the particle starting from $x_0 = 0.8$ does not reach the shock in time to experience the repulsive effect and therefore collapses directly towards $x = 0$.

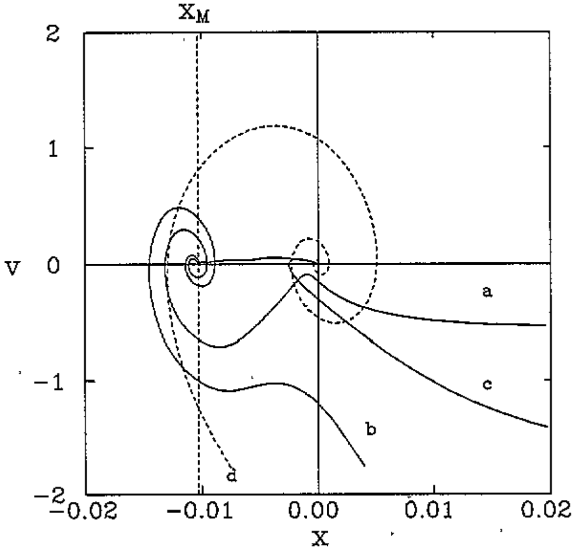

Figure 3. Trajectories in the phase plane with $\xi = 0.1$, $v_0 = 0$: line a—$x_0 = 0.05$, $a = 0.02$; line b—$x_0 = 0.4$, $a = 0.02$; line c—$x_0 = 0.8$, $a = 0.02$; line d—$x_0 = 0.4$, $a = 0.001$.

In addition, the case $a = 0.001 < a_c$ is shown in figure 3 as a further test of the simplified analysis: in this case no repulsive interaction occurs with the shock and the motion again displays a collapse towards the shock centre.

In conclusion, the numerical experiments have shown that the viscous shock centre is the final equilibrium position for any impurity, independent of its parameters and initial conditions. The detailed interaction with the flow is driven by the local features, i.e. mean slope and curvature of the velocity profile.

In particular, if the interaction occurs near the critical time, impurities with small relaxation time are rejected or not depending on their radius, according to the simplified analysis carried out for steady flow conditions. Moreover, the rejected impurities are found to attain equilibrium positions for intermediate times (of the order of the critical time itself) because of the local structure of the velocity field.

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